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A Study on Generalized Balancing Numbers

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Author's contribution

The sole author designed, analyzed, interpreted and prepared the manuscript.

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ABSTRACT

In this paper, we investigate properties of the generalized balancing sequence and we deal with, in detail, namely, balancing, modified Lucas-balancing and Lucas-balancing sequences. We present Binet's formulas, generating functions and Simson formulas for these sequences. We also present sum formulas of these sequences. We provide the proofs to indicate how the sum formulas, in general, were discovered. Of course, all the listed sum formulas may be proved by induction, but that method of proof gives no clue about their discovery. Moreover, we consider generalized balancing sequence at negative indices and construct the relationship between the sequence and itself at positive indices. This illustrates the recurrence property of the sequence at the negative index. Meanwhile, this connection holds for all integers. Furthermore, we give some identities and matrices related with these sequences.

Keywords: Balancing numbers; modified Lucas-balancing numbers; Lucas-balancing numbers; generalized Fibonacci numbers.

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1 INTRODUCTION

In [1], Behera and Panda introduced a new sequence of numbers called balancing numbers. They defined balancing numbers n as solutions of the diophantine equation

$$1+2+\cdots+(n-1) = (n+1)+(n+2)+\cdots+(n+r)$$

for some natural number r, called the balancer corresponding to n. The nth balancing number is denoted by B_n. Moreover, $C_n = \sqrt{8B_n^2+1}$ is called the nth Lucas-balancing number (see [2]). B_n and C_n satisfy the following second order linear recurrence relations

$$\begin{aligned} B_n &= 6B_{n-1} - B_{n-2}, \quad B_0 = 0, B_1 = 1, \ n \geq 0, \\ C_n &= 6C_{n-1} - C_{n-2}, \quad C_0 = 1, C_1 = 3, \ n \geq 0, \end{aligned}$$

respectively. $(B_n)_{n\geq 0}$ is the sequence $A001109$ in the OEIS [3], whereas $(B_n)_{n\geq 0}$ is the id-number $A001541$ in OEIS. Balancing and Lucas-balancing sequences has been studied by many authors and more detail can be found in the extensive literature dedicated to these sequences, see for example, [1,4,5,6,7,8,9,10,11,12, 13,2,14,15,16,17,18,19,20,21,22,23].

Generalizations of balancing numbers can be obtained in various ways (see for example [4,7,9,10,11,12,13,21]). Our generalizations of balancing numbers in section 2 are balancing in the sense of [7] but are not balancing in the sense of [4].

The purpose of this article is to generalize and investigate these interesting sequence of numbers (balancing numbers). First, we recall some properties of Fibonacci numbers and its generalizations, namely generalized Fibonacci numbers.

The Fibonacci numbers and their generalizations have many interesting properties and applications to almost every field such as architecture, nature, art, physics and engineering. The sequence of Fibonacci numbers $\{F_n\}_{n\geq 0}$ is defined by

$$F_n = F_{n-1} + F_{n-2}, \ n \geq 2, \quad F_0 = 0, \ F_1 = 1.$$

The generalization of Fibonacci sequence leads to several nice and interesting sequences. The generalized Fibonacci sequence (or generalized (r,s)-sequence or Horadam sequence or 2-step Fibonacci sequence) $\{W_n(W_0,W_1;r,s)\}_{n\geq 0}$ (or shortly $\{W_n\}_{n\geq 0}$) is defined (by Horadam [24]) as follows:

$$W_n = rW_{n-1}+sW_{n-2}, \quad W_0 = a, W_1 = b, \ n \geq 2 \tag{1.1}$$

where W_0, W_1 are arbitrary complex (or real) numbers and r, s are real numbers, see also Horadam [25,26,27] and Soykan [28].

For some specific values of a, b, r and s, it is worth presenting these special Horadam numbers in a table as a specific name. In literature, for example, the following names and notations (see Table 1) are used for the special cases of r, s and initial values.

Table 1. A few special case of generalized Fibonacci sequences

Name of sequence	$W_n(a,b;r,s)$	Binet Formula	OEIS [3]
Fibonacci	$W_n(0,1;1,1) = F_n$	$\frac{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n}{\sqrt{5}}$	A000045
Lucas	$W_n(2,1;1,1) = L_n$	$\left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n$	A000032
Pell	$W_n(0,1;2,1) = P_n$	$\frac{(1+\sqrt{2})^n - (1-\sqrt{2})^n}{2\sqrt{2}}$	A000129
Pell-Lucas	$W_n(2,2;2,1) = Q_n$	$(1+\sqrt{2})^n + (1-\sqrt{2})^n$	A002203
Jacobsthal	$W_n(0,1;1,2) = J_n$	$\frac{2^n-(-1)^n}{3}$	A001045
Jacobsthal-Lucas	$W_n(2,1;1,2) = j_n$	$2^n + (-1)^n$	A014551

Here, OEIS stands for On-line Encyclopedia of Integer Sequences.

The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = -\frac{r}{s}W_{-(n-1)} + \frac{1}{s}W_{-(n-2)}$$

for $n = 1, 2, 3, ...$ when $s \neq 0$. Therefore, recurrence (1.1) holds for all integer n.

Now we define two special cases of the sequence $\{W_n\}$. (r, s) sequence $\{G_n(0, 1; r, s)\}_{n\geq 0}$ and Lucas (r, s) sequence $\{H_n(2, r; r, s)\}_{n\geq 0}$ are defined, respectively, by the second-order recurrence relations

$$G_{n+2} = rG_{n+1} + sG_n, \quad G_0 = 0, G_1 = 1, \tag{1.2}$$

$$H_{n+2} = rH_{n+1} + sH_n, \quad H_0 = 2, H_1 = r, \tag{1.3}$$

The sequences $\{G_n\}_{n\geq 0}$, $\{H_n\}_{n\geq 0}$ and $\{E_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$\begin{aligned} G_{-n} &= -\frac{r}{s}G_{-(n-1)} + \frac{1}{s}G_{-(n-2)}, \\ H_{-n} &= -\frac{r}{s}H_{-(n-1)} + \frac{1}{s}H_{-(n-2)}, \end{aligned}$$

for $n = 1, 2, 3, ...$ respectively. Therefore, recurrences (1.2)-(1.3) hold for all integer n.

Some special cases of (r, s) sequence $\{G_n(0, 1; r, s)\}_{n\geq 0}$ and Lucas (r, s) sequence $\{H_n(2, r; r, s)\}_{n\geq 0}$ are as follows:

1. $G_n(0, 1; 1, 1) = F_n$, Fibonacci sequence,
2. $H_n(2, 1; 1, 1) = L_n$, Lucas sequence,
3. $G_n(0, 1; 2, 1) = P_n$, Pell sequence,
4. $H_n(2, 2; 2, 1) = Q_n$, Pell-Lucas sequence,
5. $G_n(0, 1; 1, 2) = J_n$, Jacobsthal sequence,
6. $H_n(2, 1; 1, 2) = j_n$, Jacobsthal-Lucas sequence.

The following theorem shows that the generalized Fibonacci sequence W_n at negative indices can be expressed by the sequence itself at positive indices.

Theorem 1.1. *For $n \in \mathbb{Z}$, for the generalized Fibonacci sequence (or generalized (r, s)-sequence or Horadam sequence or 2-step Fibonacci sequence) we have the following:*

(a)

$$\begin{aligned} W_{-n} &= (-1)^{-n-1}s^{-n}(W_n - H_nW_0) \\ &= (-1)^{n+1}s^{-n}(W_n - H_nW_0). \end{aligned}$$

(b)

$$W_{-n} = \frac{(-1)^{n+1}s^{-n}}{-W_1^2 + sW_0^2 + rW_0W_1}((2W_1 - rW_0)W_0W_{n+1} - (W_1^2 + sW_0^2)W_n).$$

Proof. For the proof, see Soykan [29, Theorem 3.2 and Theorem 3.3]. □

The following theorem presents sum formulas of generalized (r, s) numbers (generalized Fibonacci numbers).

Theorem 1.2. *Let x be a real (or complex) number. For all integers m and j, for generalized (r,s) numbers (generalized Fibonacci numbers), we have the following sum formulas:*

(a) *If $(-s)^m x^2 - xH_m + 1 \neq 0$ then*

$$\sum_{k=0}^{n} x^k W_{mk+j} = \frac{((-s)^m x - H_m)x^{n+1}W_{mn+j} + (-s)^m x^{n+1}W_{mn+j-m} + W_j - (-s)^m xW_{j-m}}{(-s)^m x^2 - xH_m + 1}. \quad (1.4)$$

(b) *If $(-s)^m x^2 - xH_m + 1 = u(x-a)(x-b) = 0$ for some $u, a, b \in \mathbb{C}$ with $u \neq 0$ and $a \neq b$, i.e., $x = a$ or $x = b$, then*

$$\sum_{k=0}^{n} x^k W_{mk+j} = \frac{(x(n+2)(-s)^m - (n+1)H_m)x^n W_{j+mn} + (-s)^m (n+1)x^n W_{mn+j-m} - (-s)^m W_{j-m}}{2(-s)^m x - H_m}.$$

(c) *If $(-s)^m x^2 - xH_m + 1 = u(x-c)^2 = 0$ for some $u, c \in \mathbb{C}$ with $u \neq 0$, i.e., $x = c$, then*

$$\sum_{k=0}^{n} x^k W_{mk+j} = \frac{(n+1)\left((-s)^m (n+2)x^n - nx^{n-1}H_m\right)W_{mn+j} + n(n+1)(-s)^m x^{n-1}W_{mn+j-m}}{2(-s)^m}.$$

Proof. It is given in Soykan [29, Theorem 4.1]. □

Note that (1.4) can be written in the following form

$$\sum_{k=1}^{n} x^k W_{mk+j} = \frac{((-s)^m x - H_m)x^{n+1}W_{mn+j} + (-s)^m x^{n+1}W_{mn+j-m} + x(H_m - (-s)^m x)W_j - (-s)^m xW_{j-m}}{(-s)^m x^2 - xH_m + 1}.$$

We give the ordinary generating function $\sum_{n=0}^{\infty} W_n x^n$ of the sequence $\{W_n\}$.

Lemma 1.3. *Suppose that $f_{W_n}(x) = \sum_{n=0}^{\infty} W_n x^n$ is the ordinary generating function of the generalized Fibonacci sequence $\{W_n\}_{n\geq 0}$. Then, $\sum_{n=0}^{\infty} W_n x^n$ is given by*

$$\sum_{n=0}^{\infty} W_n x^n = \frac{W_0 + (W_1 - rW_0)x}{1 - rx - sx^2}. \quad (1.5)$$

Proof. For a proof, see [28, Lemma 1.1]. □

1.1 Binet's Formula for the Distinct Roots Case and Single Root Case

Let α and β be two roots of the quadratic equation

$$x^2 - rx - s = 0, \quad (1.6)$$

of which the left-hand side is called the characteristic polynomial (or the characteristic equation) of the recurrence relation (1.1). The following theorem presents the Binet's formula of the sequence W_n.

Theorem 1.4. *The general term of the sequence W_n can be presented by the following Binet formula:*

$$W_n = \begin{cases} \dfrac{W_1 - \beta W_0}{\alpha - \beta}\alpha^n - \dfrac{W_1 - \alpha W_0}{\alpha - \beta}\beta^n & , \text{ if } \alpha \neq \beta \text{ (Distinct Roots Case)} \\ (nW_1 - \alpha(n-1)W_0)\alpha^{n-1} & , \text{ if } \alpha = \beta \text{ (Single Root Case)} \end{cases}$$
$$= \begin{cases} \dfrac{W_1 - \beta W_0}{\alpha - \beta}\alpha^n - \dfrac{W_1 - \alpha W_0}{\alpha - \beta}\beta^n & , \text{ if } \alpha \neq \beta \text{ (Distinct Roots Case)} \\ (nW_1 - \frac{r}{2}(n-1)W_0)\left(\frac{r}{2}\right)^{n-1} & , \text{ if } \alpha = \beta \text{ (Single Root Case)} \end{cases}.$$

Proof. For a proof, see Soykan [28] and [29]. □

The roots of characteristic equation are

$$\alpha = \frac{r+\sqrt{\Delta}}{2}, \quad \beta = \frac{r-\sqrt{\Delta}}{2}. \tag{1.7}$$

where

$$\Delta = r^2 + 4s$$

and the followings hold

$$\begin{aligned} \alpha + \beta &= r, \\ \alpha\beta &= -s, \\ (\alpha-\beta)^2 &= (\alpha+\beta)^2 - 4\alpha\beta = r^2 + 4s. \end{aligned}$$

If $\Delta = r^2 + 4s \neq 0$ then $\alpha \neq \beta$ i.e., there are distinct roots of the quadratic equation (1.6) and if $\Delta = r^2 + 4s = 0$ then $\alpha = \beta$, i.e., there is a single root of the quadratic equation (1.6).

In the case $r^2 + 4s \neq 0$ so that $\alpha \neq \beta$, for all integers n, (r,s) and Lucas (r,s) numbers (using initial conditions in Theorem 1.4) can be expressed using Binet's formulas as

$$\begin{aligned} G_n &= \frac{\alpha^n}{(\alpha-\beta)} + \frac{\beta^n}{(\beta-\alpha)}, \\ H_n &= \alpha^n + \beta^n, \end{aligned}$$

respectively. In the case $r^2 + 4s = 0$ so that $\alpha = \beta$, for all integers n, (r,s) and Lucas (r,s) numbers (using initial conditions in Theorem 1.4) can be expressed using Binet's formulas as

$$\begin{aligned} G_n &= n\alpha^{n-1}, \\ H_n &= 2\alpha^n, \end{aligned}$$

respectively.

2 GENERALIZED BALANCING SEQUENCE

In this paper, we consider the case $r = 6, s = -1$. A generalized balancing sequence $\{W_n\}_{n\geq 0} = \{W_n(W_0, W_1)\}_{n\geq 0}$ is defined by the second-order recurrence relation

$$W_n = 6W_{n-1} - W_{n-2} \tag{2.1}$$

with the initial values $W_0 = c_0, W_1 = c_1$ not all being zero.

The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = 6W_{-(n-1)} - W_{-(n-2)}$$

for $n = 1, 2, 3, ...$ Therefore, recurrence (2.1) holds for all integer n.

By Theorem 1.4, the Binet formula of generalized balancing numbers can be written as

$$W_n = \frac{W_1 - \beta W_0}{\alpha-\beta}\alpha^n - \frac{W_1 - \alpha W_0}{\alpha-\beta}\beta^n$$

where α and β are the roots of the quadratic equation $x^2 - 6x + 1 = 0$.Moreover

$$\begin{aligned} \alpha &= 3+2\sqrt{2} \\ \beta &= 3-2\sqrt{2} \end{aligned}$$

Note that

$$\begin{aligned} \alpha+\beta &= 6, \\ \alpha\beta &= -1, \\ \alpha-\beta &= 4\sqrt{2}. \end{aligned}$$

So

$$W_n = \frac{W_1-(3-2\sqrt{2})W_0}{4\sqrt{2}}(3+2\sqrt{2})^n - \frac{W_1-(3+2\sqrt{2})W_0}{4\sqrt{2}}(3-2\sqrt{2})^n. \tag{2.2}$$

The first few generalized balancing numbers with positive subscript and negative subscript are given in the following Table 2.

Table 2. A few generalized balancing numbers

n	W_n	W_{-n}
0	W_0	W_0
1	W_1	$6W_0 - W_1$
2	$6W_1 - W_0$	$35W_0 - 6W_1$
3	$35W_1 - 6W_0$	$204W_0 - 35W_1$
4	$204W_1 - 35W_0$	$1189W_0 - 204W_1$
5	$1189W_1 - 204W_0$	$6930W_0 - 1189W_1$
6	$6930W_1 - 1189W_0$	$40391W_0 - 6930W_1$
7	$40391W_1 - 6930W_0$	$235416W_0 - 40391W_1$
8	$235416W_1 - 40391W_0$	$1372105W_0 - 235416W_1$
9	$1372105W_1 - 235416W_0$	$7997214W_0 - 1372105W_1$
10	$7997214W_1 - 1372105W_0$	$46611179W_0 - 7997214W_1$

Now we define three special cases of the sequence $\{W_n\}$. balancing sequence $\{B_n\}_{n\geq0}$, modified Lucas-balancing sequence $\{H_n\}_{n\geq0}$ and Lucas-balancing sequence $\{C_n\}_{n\geq0}$ are defined, respectively, by the second-order recurrence relations

$$B_n = 6B_{n-1} - B_{n-2}, \quad B_0 = 0, B_1 = 1, \tag{2.3}$$
$$H_n = 6H_{n-1} - H_{n-2}, \quad H_0 = 2, H_1 = 6, \tag{2.4}$$
$$C_n = 6C_{n-1} - C_{n-2}, \quad C_0 = 1, C_1 = 3. \tag{2.5}$$

The sequences $\{B_n\}_{n\geq0}$, $\{H_n\}_{n\geq0}$ and $\{C_n\}_{n\geq0}$ can be extended to negative subscripts by defining

$$\begin{aligned} B_{-n} &= 6B_{-(n-1)} - B_{-(n-2)}, \\ H_{-n} &= 6H_{-(n-1)} - H_{-(n-2)}, \\ C_{-n} &= 6C_{-(n-1)} - C_{-(n-2)}, \end{aligned}$$

for $n = 1, 2, 3, ...$ respectively. Therefore, recurrences (2.3)-(2.5) hold for all integer n.

Next, we present the first few values of the balancing, modified Lucas-balancing and Lucas-balancing numbers with positive and negative subscripts:

Table 3. The first few values of the special second-order numbers with positive and negative subscripts

n	0	1	2	3	4	5	6	7	8	9	10	11
B_n	0	1	6	35	204	1189	6930	40391	235416	1372105	7997214	46 611179
B_{-n}		−1	−6	−35	−204	−1189	−6930	−40391	−235416	−1372105	−7997214	−46 611179
H_n	2	6	34	198	1154	6726	39202	228486	1331714	7761798	45239074	263672646
H_{-n}		6	34	198	1154	6726	39202	228486	1331714	7761798	45239074	263672646
C_n	1	3	17	99	577	3363	19601	114243	665857	3880899	22619537	131836 323
C_{-n}		3	17	99	577	3363	19 601	114243	665857	3880899	22619537	131836323

For all integers n, balancing, modified Lucas-balancing and Lucas-balancing numbers (using initial conditions in Theorem 1.4) can be expressed using Binet's formulas as

$$\begin{aligned} B_n &= \frac{\alpha^n}{(\alpha-\beta)} + \frac{\beta^n}{(\beta-\alpha)}, \\ H_n &= \alpha^n + \beta^n, \\ C_n &= \frac{\alpha^n + \beta^n}{2}, \end{aligned}$$

respectively. Note that

$$C_n = \frac{H_n}{2}.$$

Next, we give the ordinary generating function $\sum\limits_{n=0}^{\infty} W_n x^n$ of the sequence $\{W_n\}$.

Lemma 2.1. *Suppose that $f_{W_n}(x) = \sum\limits_{n=0}^{\infty} W_n x^n$ is the ordinary generating function of the generalized balancing sequence $\{W_n\}_{n\geq 0}$. Then, $\sum\limits_{n=0}^{\infty} W_n x^n$ is given by*

$$\sum_{n=0}^{\infty} W_n x^n = \frac{W_0 + (W_1 - 6W_0)x}{1 - 6x + x^2} \tag{2.6}$$

Proof. In Lemma 1.3, take $r = 6, s = -1$. □

The previous Lemma gives the following results as particular examples.

Corollary 2.2. *Generated functions of balancing, modified Lucas-balancing and Lucas-balancing numbers are*

$$\begin{aligned} \sum_{n=0}^{\infty} B_n x^n &= \frac{x}{1-6x+x^2}, \\ \sum_{n=0}^{\infty} H_n x^n &= \frac{2-6x}{1-6x+x^2}, \\ \sum_{n=0}^{\infty} C_n x^n &= \frac{1-3x}{1-6x+x^2}, \end{aligned}$$

respectively.

Proof. In Lemma 2.1, take $W_n = B_n$ with $B_0 = 0, B_1 = 1$, $W_n = H_n$ with $H_0 = 2, H_1 = 6$ and $W_n = C_n$ with $C_0 = 1, C_1 = 3$, respectively. □

3 SIMSON FORMULAS

There is a well-known Simson Identity (formula) for Fibonacci sequence $\{F_n\}$, namely,

$$F_{n+1}F_{n-1} - F_n^2 = (-1)^n$$

which was derived first by R. Simson in 1753 and it is now called as Cassini Identity (formula) as well. This can be written in the form

$$\begin{vmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{vmatrix} = (-1)^n.$$

The following theorem gives generalization of this result to the generalized balancing sequence $\{W_n\}_{n\geq 0}$.

Theorem 3.1 (Simson Formula of Generalized Balancing Numbers)**.** *For all integers n, we have*

$$\begin{vmatrix} W_{n+1} & W_n \\ W_n & W_{n-1} \end{vmatrix} = \begin{vmatrix} W_1 & W_0 \\ W_0 & W_{-1} \end{vmatrix} = -(W_1^2 + W_0^2 - 6W_0W_1). \tag{3.1}$$

Proof. For a proof of Eq. (3.1), see Soykan [30], just take $s = -1$. □

The previous theorem gives the following results as particular examples.

Corollary 3.2. *For all integers n, balancing, modified Lucas-balancing and Lucas-balancing numbers are given as*

$$\begin{aligned}
\begin{vmatrix} B_{n+1} & B_n \\ B_n & B_{n-1} \end{vmatrix} &= -1, \\
\begin{vmatrix} H_{n+1} & H_n \\ H_n & H_{n-1} \end{vmatrix} &= 32, \\
\begin{vmatrix} C_{n+1} & C_n \\ C_n & C_{n-1} \end{vmatrix} &= 8.
\end{aligned}$$

respectively.

4 SOME IDENTITIES

In this section, we obtain some identities of generalized balancing, balancing, modified Lucas-balancing and Lucas-balancing numbers. First, we can give a few basic relations between $\{W_n\}$ and $\{B_n\}$.

Lemma 4.1. *The following equalities are true:*

$$\begin{aligned}
W_n &= (204W_0 - 35W_1)B_{n+4} + (-1189W_0 + 204W_1)B_{n+3}, \qquad (4.1)\\
W_n &= (35W_0 - 6W_1)B_{n+3} + (-204W_0 + 35W_1)B_{n+2}, \\
W_n &= (6W_0 - W_1)B_{n+2} + (-35W_0 + 6W_1)B_{n+1}, \\
W_n &= W_0B_{n+1} + (W_1 - 6W_0)B_n, \\
W_n &= W_1B_n - W_0B_{n-1},
\end{aligned}$$

and

$$\begin{aligned}
(W_0^2 + W_1^2 - 6W_0W_1)B_n &= (6W_0 - 35W_1)W_{n+4} - (35W_0 - 204W_1)W_{n+3}, \\
(W_0^2 + W_1^2 - 6W_0W_1)B_n &= (W_0 - 6W_1)W_{n+3} - (6W_0 - 35W_1)W_{n+2}, \\
(W_0^2 + W_1^2 - 6W_0W_1)B_n &= -W_1W_{n+2} - (W_0 - 6W_1)W_{n+1}, \\
(W_0^2 + W_1^2 - 6W_0W_1)B_n &= -W_0W_{n+1} + W_1W_n, \\
(W_0^2 + W_1^2 - 6W_0W_1)B_n &= (W_1 - 6W_0)W_n + W_0W_{n-1}.
\end{aligned}$$

Proof. Note that all the identities hold for all integers n. We prove (4.1). To show (4.1), writing

$$W_n = a \times B_{n+4} + b \times B_{n+3}$$

and solving the system of equations

$$\begin{aligned} W_0 &= a \times B_4 + b \times B_3 \\ W_1 &= a \times B_5 + b \times B_4 \end{aligned}$$

we find that $a = 204W_0 - 35W_1, b = -1189W_0 + 204W_1$. The other equalities can be proved similarly. □

Note that all the identities in the above Lemma can be proved by induction as well.

Next, we present a few basic relations between $\{H_n\}$ and $\{W_n\}$.

Lemma 4.2. *The following equalities are true:*

$$\begin{aligned} 16W_n &= -(577W_0 - 99W_1)H_{n+4} + (3363W_0 - 577W_1)H_{n+3}, \\ 16W_n &= -(99W_0 - 17W_1)H_{n+3} + (577W_0 - 99W_1)H_{n+2}, \\ 16W_n &= -(17W_0 - 3W_1)H_{n+2} + (99W_0 - 17W_1)H_{n+1}, \\ 16W_n &= -(3W_0 - W_1)H_{n+1} + (17W_0 - 3W_1)H_n, \\ 16W_n &= -(W_0 - 3W_1)H_n + (3W_0 - W_1)H_{n-1}, \end{aligned}$$

and

$$\begin{aligned} (W_0^2 + W_1^2 - 6W_0W_1)H_n &= -2(17W_0 - 99W_1)W_{n+4} + 2(99W_0 - 577W_1)W_{n+3}, \\ (W_0^2 + W_1^2 - 6W_0W_1)H_n &= -2(3W_0 - 17W_1)W_{n+3} + 2(17W_0 - 99W_1)W_{n+2}, \\ (W_0^2 + W_1^2 - 6W_0W_1)H_n &= -2(W_0 - 3W_1)W_{n+2} + 2(3W_0 - 17W_1)W_{n+1}, \\ (W_0^2 + W_1^2 - 6W_0W_1)H_n &= -2(3W_0 - W_1)W_{n+1} + 2(W_0 - 3W_1)W_n, \\ (W_0^2 + W_1^2 - 6W_0W_1)H_n &= -2(17W_0 - 3W_1)W_n + 2(3W_0 - W_1)W_{n-1}. \end{aligned}$$

Now, we give a few basic relations between $\{W_n\}$ and $\{C_n\}$.

Lemma 4.3. *The following equalities are true:*

$$\begin{aligned} 8W_n &= -(577W_0 - 99W_1)C_{n+4} + (3363W_0 - 577W_1)C_{n+3}, \\ 8W_n &= -(99W_0 - 17W_1)C_{n+3} + (577W_0 - 99W_1)C_{n+2}, \\ 8W_n &= -(17W_0 - 3W_1)C_{n+2} + (99W_0 - 17W_1)C_{n+1}, \\ 8W_n &= -(3W_0 - W_1)C_{n+1} + (17W_0 - 3W_1)C_n, \\ 8W_n &= -(W_0 - 3W_1)C_n + (3W_0 - W_1)C_{n-1}, \end{aligned}$$

and

$$\begin{aligned} (W_0^2 + W_1^2 - 6W_0W_1)C_n &= -(17W_0 - 99W_1)W_{n+4} + (99W_0 - 577W_1)W_{n+3}, \\ (W_0^2 + W_1^2 - 6W_0W_1)C_n &= -(3W_0 - 17W_1)W_{n+3} + (17W_0 - 99W_1)W_{n+2}, \\ (W_0^2 + W_1^2 - 6W_0W_1)C_n &= -(W_0 - 3W_1)W_{n+2} + (3W_0 - 17W_1)W_{n+1}, \\ (W_0^2 + W_1^2 - 6W_0W_1)C_n &= (-3W_0 + W_1)W_{n+1} + (W_0 - 3W_1)W_n, \\ (W_0^2 + W_1^2 - 6W_0W_1)C_n &= (-17W_0 + 3W_1)W_n + (3W_0 - W_1)W_{n-1}. \end{aligned}$$

Next, we present a few basic relations between $\{B_n\}$ and $\{H_n\}$.

Lemma 4.4. *The following equalities are true:*

$$\begin{aligned} H_n &= 198B_{n+4} - 1154B_{n+3}, \\ H_n &= 34B_{n+3} - 198B_{n+2}, \\ H_n &= 6B_{n+2} - 34B_{n+1}, \\ H_n &= 2B_{n+1} - 6B_n, \\ H_n &= 6B_n - 2B_{n-1}, \end{aligned}$$

and

$$\begin{aligned} 32B_n &= 198H_{n+4} - 1154H_{n+3}, \\ 32B_n &= 34H_{n+3} - 198H_{n+2}, \\ 32B_n &= 6H_{n+2} - 34H_{n+1}, \\ 32B_n &= 2H_{n+1} - 6H_n, \\ 32B_n &= 6H_n - 2H_{n-1}. \end{aligned}$$

Now, we give a few basic relations between $\{B_n\}$ and $\{C_n\}$.

Lemma 4.5. *The following equalities are true:*

$$\begin{aligned} C_n &= 99B_{n+4} - 577B_{n+3}, \\ C_n &= 17B_{n+3} - 99B_{n+2}, \\ C_n &= 3B_{n+2} - 17B_{n+1}, \\ C_n &= B_{n+1} - 3B_n, \\ C_n &= 3B_n - B_{n-1}, \end{aligned}$$

and

$$\begin{aligned} 8B_n &= 99C_{n+4} - 577C_{n+3}, \\ 8B_n &= 17C_{n+3} - 99C_{n+2}, \\ 8B_n &= 3C_{n+2} - 17C_{n+1}, \\ 8B_n &= C_{n+1} - 3C_n, \\ 8B_n &= 3C_n - C_{n-1}. \end{aligned}$$

Next, we present a few basic relations between $\{H_n\}$ and $\{C_n\}$.

Lemma 4.6. *The following equalities are true:*

$$\begin{aligned} H_n &= -70C_{n+4} + 408C_{n+3}, \\ H_n &= -12C_{n+3} + 70C_{n+2}, \\ H_n &= -2C_{n+2} + 12C_{n+1}, \\ H_n &= 2C_n, \end{aligned}$$

and

$$\begin{aligned} 2C_n &= -35H_{n+4} + 204H_{n+3}, \\ 2C_n &= -6H_{n+3} + 35H_{n+2}, \\ 2C_n &= -H_{n+2} + 6H_{n+1}, \\ 2C_n &= H_n. \end{aligned}$$

We now present a few special identities for the generalized balancing sequence $\{W_n\}$.

Theorem 4.7. *(Catalan's identity of the generalized balancing sequence) For all integers n and m, the following identity holds:*

$$W_{n+m}W_{n-m} - W_n^2 = -\frac{1}{32}(W_1^2 + W_0^2 - 6W_0W_1)\left(\left(-2\sqrt{2}+3\right)^m - \left(2\sqrt{2}+3\right)^m\right)^2.$$

Proof. We use the identity (2.2). □

As special cases of the above theorem, we have the following corollary.

Corollary 4.8. *For all integers n and m, the following identities hold:*

(a) $B_{n+m}B_{n-m} - B_n^2 = -\frac{1}{32}\left(\left(-2\sqrt{2}+3\right)^m - \left(2\sqrt{2}+3\right)^m\right)^2.$

(b) $H_{n+m}H_{n-m} - H_n^2 = \left(\left(-2\sqrt{2}+3\right)^m - \left(2\sqrt{2}+3\right)^m\right)^2.$

(c) $C_{n+m}C_{n-m} - C_n^2 = \frac{1}{4}\left(\left(-2\sqrt{2}+3\right)^m - \left(2\sqrt{2}+3\right)^m\right)^2.$

Note that for $m = 1$ in Catalan's identity of the generalized balancing sequence, we get the Cassini identity for the generalized balancing sequnce.

Theorem 4.9. *(Cassini's identity of the generalized balancing sequence) For all integers n, the following identity holds:*

$$W_{n+1}W_{n-1} - W_n^2 = -(W_1^2 + W_0^2 - 6W_0W_1).$$

As special cases of the above theorem, we have the following corollary.

Corollary 4.10. *For all integers n, the following identities hold:*

(a) $B_{n+1}B_{n-1} - B_n^2 = -1.$

(b) $H_{n+1}H_{n-1} - H_n^2 = 32.$

(c) $C_{n+1}C_{n-1} - C_n^2 = 8.$

The d'Ocagne's, Gelin-Cesàro's and Melham' identities can also be obtained by using (2.2). The next theorem presents d'Ocagne's, Gelin-Cesàro's and Melham' identities of generalized balancing sequence $\{W_n\}$.

Theorem 4.11. *Let n and m be any integers. Then the following identities are true:*

(a) *(d'Ocagne's identity)*
$W_{m+1}W_n - W_mW_{n+1} = \frac{1}{8}\sqrt{2}(W_1^2 + W_0^2 - 6W_0W_1)((-2\sqrt{2}+3)^m(2\sqrt{2}+3)^n - (-2\sqrt{2}+3)^n(2\sqrt{2}+3)^m).$

(b) *(Gelin-Cesàro's identity)*
$W_{n+2}W_{n+1}W_{n-1}W_{n-2} - W_n^4 = \frac{37}{32}(W_1^2 + W_0^2 - 6W_0W_1)(-((3-2\sqrt{2})^{2n} + (3+2\sqrt{2})^{2n} - \frac{1226}{37})W_1^2 - ((3-2\sqrt{2})^{2n}(17+12\sqrt{2}) - (3+2\sqrt{2})^{2n}(-17+12\sqrt{2}) - \frac{1226}{37})W_0^2 + 2((3-2\sqrt{2})^{2n}(3+2\sqrt{2}) - (-3+2\sqrt{2})(3+2\sqrt{2})^{2n} - \frac{3678}{37})W_0W_1).$

(c) *(Melham's identity)*
$W_{n+1}W_{n+2}W_{n+6} - W_{n+3}^3 = -\frac{1}{8}(W_1^2 + W_0^2 - 6W_0W_1)(-((3-2\sqrt{2})^n(-5180+3669\sqrt{2}) - (3+2\sqrt{2})^n(5180+3669\sqrt{2}))W_1 + ((3-2\sqrt{2})^n(-864+647\sqrt{2}) - (3+2\sqrt{2})^n(864+647\sqrt{2}))W_0).$

Proof. Use the identity (2.2). □

As special cases of the above theorem, we have the following three corollaries. First one presents d'Ocagne's, Gelin-Cesàro's and Melham' identities of balancing sequence $\{B_n\}$.

Corollary 4.12. *Let n and m be any integers. Then the following identities are true:*

(a) *(d'Ocagne's identity)*

$$B_{m+1}B_n - B_mB_{n+1} = \frac{1}{8}\sqrt{2}((-2\sqrt{2}+3)^m(2\sqrt{2}+3)^n - (-2\sqrt{2}+3)^n(2\sqrt{2}+3)^m).$$

(b) *(Gelin-Cesàro's identity)*

$$B_{n+2}B_{n+1}B_{n-1}B_{n-2} - B_n^4 = -\frac{37}{32}((3-2\sqrt{2})^{2n} + (3+2\sqrt{2})^{2n} - \frac{1226}{37}).$$

(c) *(Melham's identity)*

$$B_{n+1}B_{n+2}B_{n+6} - B_{n+3}^3 = \frac{1}{8}((3-2\sqrt{2})^n(-5180+3669\sqrt{2}) - \left((3+2\sqrt{2})\right)^n(5180+3669\sqrt{2})).$$

Second one presents d'Ocagne's, Gelin-Cesàro's and Melham' identities of modified Lucas-balancing sequence $\{H_n\}$.

Corollary 4.13. *Let n and m be any integers. Then the following identities are true:*

(a) *(d'Ocagne's identity)*

$$H_{m+1}H_n - H_mH_{n+1} = -4\sqrt{2}((-2\sqrt{2}+3)^m(2\sqrt{2}+3)^n - (-2\sqrt{2}+3)^n(2\sqrt{2}+3)^m).$$

(b) *(Gelin-Cesàro's identity)*

$$H_{n+2}H_{n+1}H_{n-1}H_{n-2} - H_n^4 = 1184((3-2\sqrt{2})^{2n} + (3+2\sqrt{2})^{2n} + \frac{1226}{37}).$$

(c) *(Melham's identity)*

$$H_{n+1}H_{n+2}H_{n+6} - H_{n+3}^3 = 32(-(3-2\sqrt{2})^n(-3669+2590\sqrt{2}) + (3+2\sqrt{2})^n(3669+2590\sqrt{2})).$$

Third one presents d'Ocagne's, Gelin-Cesàro's and Melham' identities of balancing sequence $\{C_n\}$.

Corollary 4.14. *Let n and m be any integers. Then the following identities are true:*

(a) *(d'Ocagne's identity)*

$$C_{m+1}C_n - C_mC_{n+1} = -\sqrt{2}((-2\sqrt{2}+3)^m(2\sqrt{2}+3)^n - (-2\sqrt{2}+3)^n(2\sqrt{2}+3)^m)$$

(b) *(Gelin-Cesàro's identity)*

$$C_{n+2}C_{n+1}C_{n-1}C_{n-2} - C_n^4 = 74((3-2\sqrt{2})^{2n} + (3+2\sqrt{2})^{2n} + \frac{1226}{37})$$

(c) *(Melham's identity)*

$$C_{n+1}C_{n+2}C_{n+6} - C_{n+3}^3 = 4(-(3-2\sqrt{2})^n(-3669+2590\sqrt{2}) + (3+2\sqrt{2})^n(3669+2590\sqrt{2}))$$

5 ON THE RECURRENCE PROPERTIES OF GENERALIZED BALANCING SEQUENCE

Taking $r = 6, s = -1$ in Theorem 1.1 (a) and (b), we obtain the following Proposition.

Proposition 5.1. *For $n \in \mathbb{Z}$, generalized balancing numbers (the case $r = 6, s = -1$) have the following identity:*

$$\begin{aligned} W_{-n} &= -(W_n - H_n W_0) \\ &= \frac{-1}{-W_1^2 - W_0^2 + 6W_0W_1}((2W_1 - 6W_0)W_0W_{n+1} - (W_1^2 - W_0^2)W_n). \end{aligned}$$

From the above Proposition, we have the following corollary which gives the connection between the special cases of generalized balancing sequence at the positive index and the negative index: for balancing, modified Lucas-balancing and Lucas-balancing numbers: take $W_n = B_n$ with $B_0 = 0, B_1 = 1$, take $W_n = H_n$ with $H_0 = 2, H_1 = 6$ and $W_n = C_n$ with $C_0 = 1, C_1 = 3$, respectively. Note that in this case $H_n = H_n$.

Corollary 5.2. *For $n \in \mathbb{Z}$, we have the following recurrence relations:*

(a) *balancing sequence:*

$$B_{-n} = -B_n = -\left(\frac{\alpha^n}{(\alpha-\beta)} + \frac{\beta^n}{(\beta-\alpha)}\right).$$

(b) *modified Lucas-balancing sequence:*

$$H_{-n} = H_n = \alpha^n + \beta^n.$$

(c) *Lucas-balancing sequence:*

$$C_{-n} = C_n = \frac{\alpha^n + \beta^n}{2}.$$

6 THE SUM FORMULA $\sum_{k=0}^{n} x^k W_{mk+j}$

The following theorem presents sum formulas of generalized balancing numbers.

Theorem 6.1. *Let x be a real (or complex) number. For all integers m and j, for generalized balancing numbers we have the following sum formulas:*

(a) *if $x^2 - xH_m + 1 \neq 0$ then*

$$\sum_{k=0}^{n} x^k W_{mk+j} = \frac{(x - H_m)x^{n+1}W_{mn+j} + x^{n+1}W_{mn+j-m} + W_j - xW_{j-m}}{x^2 - xH_m + 1} \tag{6.1}$$

(b) *If $x^2 - xH_m + 1 = (x-a)(x-b) = 0$ for some $u, a, b \in \mathbb{C}$ with $u \neq 0$ and $a \neq b$, i.e., $x = a$ or $x = b$, then*

$$\sum_{k=0}^{n} x^k W_{mk+j} = \frac{((2+n)x - (n+1)H_m)x^n W_{mn+j} + (n+1)x^n W_{mn+j-m} - W_{j-m}}{2x - H_m}$$

(c) *If $x^2 - xH_m + 1 = (x-c)^2 = 0$ for some $u, c \in \mathbb{C}$ with $u \neq 0$, i.e., $x = c$, then*

$$\sum_{k=0}^{n} x^k W_{mk+j} = \frac{(n+1)((n+2)x^n - nx^{n-1}H_m)W_{mn+j} + n(n+1)x^{n-1}W_{mn+j-m}}{2}.$$

Proof. Take $r = 6, s = -1$ and $H_n = H_n$ in Theorem 1.2. □

Note that (6.1) can be written in the following form

$$\sum_{k=1}^{n} x^k W_{mk+j} = \frac{(x - H_m)x^{n+1}W_{mn+j} + x^{n+1}W_{mn+j-m} + x(H_m - x)W_j - xW_{j-m}}{x^2 - xH_m + 1}.$$

As special cases of m and j in the last Theorem, we obtain the following proposition.

Proposition 6.2. *For generalized balancing numbers (the case $r = 6, s = -1$) we have the following sum formulas:*

(a) $(m = 1, j = 0)$
If $x^2 - 6x + 1 \neq 0$, *i.e.,* $x \neq 3 + 2\sqrt{2}, x \neq 3 - 2\sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k W_k = \frac{(x - 6)x^{n+1}W_n + x^{n+1}W_{n-1} + (W_1 - 6W_0)x + W_0}{x^2 - 6x + 1},$$

and
if $x^2 - 6x + 1 = 0$, *i.e.,* $x = 3 + 2\sqrt{2}$ *or* $x = 3 - 2\sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k W_k = \frac{(2x - 6 + n(x - 6))x^n W_n + (n + 1)x^n W_{n-1} + (W_1 - 6W_0)}{2x - 6}.$$

(b) $(m = 2, j = 0)$
If $x^2 - 34x + 1 \neq 0$, *i.e.,* $x \neq 17 + 12\sqrt{2}, x \neq 17 - 12\sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k W_{2k} = \frac{(x - 34)x^{n+1}W_{2n} + x^{n+1}W_{2n-2} + (6W_1 - 35W_0)x + W_0}{x^2 - 34x + 1},$$

and
if $x^2 - 34x + 1 = 0$, *i.e.,* $x = 17 + 12\sqrt{2}$ *or* $x = 17 - 12\sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k W_{2k} = \frac{(2x - 34 + n(x - 34))x^n W_{2n} + (n + 1)x^n W_{2n-2} + (6W_1 - 35W_0)}{2x - 34}.$$

(c) $(m = 2, j = 1)$
If $x^2 - 34x + 1 \neq 0$, *i.e.,* $x \neq 17 + 12\sqrt{2}, x \neq 17 - 12\sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k W_{2k+1} = \frac{(x - 34)x^{n+1}W_{2n+1} + x^{n+1}W_{2n-1} + (W_1 - 6W_0)x + W_1}{x^2 - 34x + 1},$$

and
if $x^2 - 34x + 1 = 0$, *i.e.,* $x = 17 + 12\sqrt{2}$ *or* $x = 17 - 12\sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k W_{2k+1} = \frac{((2 + n)x - 34(n + 1))x^n W_{2n+1} + (n + 1)x^n W_{2n-1} + (W_1 - 6W_0)}{2x - 34}.$$

(d) $(m = -1, j = 0)$
If $x^2 - 6x + 1 \neq 0$, *i.e.,* $x \neq 3 + 2\sqrt{2}, x \neq 3 - 2\sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k W_{-k} = \frac{x^{n+1}W_{-n+1} + (x - 6)x^{n+1}W_{-n} - W_1 x + W_0}{x^2 - 6x + 1},$$

and
if $x^2-6x+1=0$, *i.e.,* $x=3+2\sqrt{2}$ *or* $x=3-2\sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k W_{-k} = \frac{(n+1)x^n W_{-n+1}+(2x-6+n(x-6))x^n W_{-n}-W_1}{2x-6}.$$

(e) $(m=-2, j=0)$
If $x^2-34x+1\neq 0$, *i.e.,* $x\neq 17+12\sqrt{2}$, $x\neq 17-12\sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k W_{-2k} = \frac{x^{n+1}W_{-2n+2}+(x-34)x^{n+1}W_{-2n}-xW_2+W_0}{x^2-34x+1},$$

and
if $x^2-34x+1=0$, *i.e.,* $x=17+12\sqrt{2}$ *or* $x=17-12\sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k W_{-2k} = \frac{(n+1)x^n W_{-2n+2}+(2x-34+n(x-34))x^n W_{-2n}-W_2}{2x-34}.$$

(f) $(m=-2, j=1)$
If $x^2-34x+1\neq 0$, *i.e.,* $x\neq 17+12\sqrt{2}$, $x\neq 17-12\sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k W_{-2k+1} = \frac{x^{n+1}W_{-2n+3}+(x-34)x^{n+1}W_{-2n+1}-W_3x+W_1}{x^2-34x+1},$$

and
if $x^2-34x+1=0$, *i.e.,* $x=17+12\sqrt{2}$ *or* $x=17-12\sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k W_{-2k+1} = \frac{(n+1)x^n W_{-2n+3}+(2x-34+n(x-34))x^n W_{-2n+1}-W_3}{2x-34}.$$

From the above proposition, we have the following corollary which gives sum formulas of balancing numbers (take $W_n=B_n$ with $B_0=.0, B_1=1$).

Corollary 6.3. *For* $n\geq 0$, *balancing numbers have the following properties:*

(a) $(m=1, j=0)$
If $x^2-6x+1\neq 0$, *i.e.,* $x\neq 3+2\sqrt{2}$, $x\neq 3-2\sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k B_k = \frac{(x-6)x^{n+1}B_n+x^{n+1}B_{n-1}+x}{x^2-6x+1},$$

and
if $x^2-6x+1=0$, *i.e.,* $x=3+2\sqrt{2}$ *or* $x=3-2\sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k B_k = \frac{(2x-6+n(x-6))x^n B_n+(n+1)x^n B_{n-1}+1}{2x-6}.$$

(b) $(m=2, j=0)$
If $x^2-34x+1\neq 0$, *i.e.,* $x\neq 17+12\sqrt{2}$, $x\neq 17-12\sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k B_{2k} = \frac{(x-34)x^{n+1}B_{2n}+x^{n+1}B_{2n-2}+6x}{x^2-34x+1},$$

and
if $x^2-34x+1=0$, *i.e.,* $x=17+12\sqrt{2}$ *or* $x=17-12\sqrt{2}$, *then*

$$\sum_{k=0}^{n}x^kB_{2k}=\frac{(2x-34+n(x-34))x^nB_{2n}+(n+1)x^nB_{2n-2}+6}{2x-34}.$$

(c) $(m=2,\ j=1)$
If $x^2-34x+1\neq 0$, *i.e.,* $x\neq 17+12\sqrt{2}$, $x\neq 17-12\sqrt{2}$, *then*

$$\sum_{k=0}^{n}x^kB_{2k+1}=\frac{(x-34)x^{n+1}B_{2n+1}+x^{n+1}B_{2n-1}+x+1}{x^2-34x+1},$$

and
if $x^2-34x+1=0$, *i.e.,* $x=17+12\sqrt{2}$ *or* $x=17-12\sqrt{2}$, *then*

$$\sum_{k=0}^{n}x^kB_{2k+1}=\frac{((2+n)x-34(n+1))x^nB_{2n+1}+(n+1)x^nB_{2n-1}+1}{2x-34}.$$

(d) $(m=-1,\ j=0)$
If $x^2-6x+1\neq 0$, *i.e.,* $x\neq 3+2\sqrt{2}$, $x\neq 3-2\sqrt{2}$, *then*

$$\sum_{k=0}^{n}x^kB_{-k}=\frac{x^{n+1}B_{-n+1}+(x-6)x^{n+1}B_{-n}-x}{x^2-6x+1},$$

and
if $x^2-6x+1=0$, *i.e.,* $x=3+2\sqrt{2}$ *or* $x=3-2\sqrt{2}$, *then*

$$\sum_{k=0}^{n}x^kB_{-k}=\frac{(n+1)x^nB_{-n+1}+(2x-6+n(x-6))x^nB_{-n}-1}{2x-6}.$$

(e) $(m=-2,\ j=0)$
If $x^2-34x+1\neq 0$, *i.e.,* $x\neq 17+12\sqrt{2}$, $x\neq 17-12\sqrt{2}$, *then*

$$\sum_{k=0}^{n}x^kB_{-2k}=\frac{x^{n+1}B_{-2n+2}+(x-34)x^{n+1}B_{-2n}-6x}{x^2-34x+1},$$

and
if $x^2-34x+1=0$, *i.e.,* $x=17+12\sqrt{2}$ *or* $x=17-12\sqrt{2}$, *then*

$$\sum_{k=0}^{n}x^kB_{-2k}=\frac{(n+1)x^nB_{-2n+2}+(2x-34+n(x-34))x^nB_{-2n}-6}{2x-34}.$$

(f) $(m=-2,\ j=1)$
If $x^2-34x+1\neq 0$, *i.e.,* $x\neq 17+12\sqrt{2}$, $x\neq 17-12\sqrt{2}$, *then*

$$\sum_{k=0}^{n}x^kB_{-2k+1}=\frac{x^{n+1}B_{-2n+3}+(x-34)x^{n+1}B_{-2n+1}-35x+1}{x^2-34x+1},$$

and
if $x^2-34x+1=0$, *i.e.,* $x=17+12\sqrt{2}$ *or* $x=17-12\sqrt{2}$, *then*

$$\sum_{k=0}^{n}x^kB_{-2k+1}=\frac{(n+1)x^nB_{-2n+3}+(2x-34+n(x-34))x^nB_{-2n+1}-35}{2x-34}.$$

Taking $W_n = H_n$ with $H_0 = 2, H_1 = 6$ in the last proposition, we have the following corollary which presents sum formulas of modified Lucas-balancing numbers.

Corollary 6.4. *For $n \geq 0,$ modified Lucas-balancing numbers have the following properties:*

(a) $(m = 1, j = 0)$

If $x^2 - 6x + 1 \neq 0,$ i.e., $x \neq 3 + 2\sqrt{2}, x \neq 3 - 2\sqrt{2},$ then

$$\sum_{k=0}^{n} x^k H_k = \frac{(x-6)x^{n+1}H_n + x^{n+1}H_{n-1} - 6x + 2}{x^2 - 6x + 1},$$

and

if $x^2 - 6x + 1 = 0,$ i.e., $x = 3 + 2\sqrt{2}$ or $x = 3 - 2\sqrt{2},$ then

$$\sum_{k=0}^{n} x^k H_k = \frac{(2x - 6 + n(x-6))x^n H_n + (n+1)x^n H_{n-1} - 6}{2x - 6}.$$

(b) $(m = 2, j = 0)$

If $x^2 - 34x + 1 \neq 0,$ i.e., $x \neq 17 + 12\sqrt{2}, x \neq 17 - 12\sqrt{2},$ then

$$\sum_{k=0}^{n} x^k H_{2k} = \frac{(x-34)x^{n+1}H_{2n} + x^{n+1}H_{2n-2} - 34x + 2}{x^2 - 34x + 1},$$

and

if $x^2 - 34x + 1 = 0,$ i.e., $x = 17 + 12\sqrt{2}$ or $x = 17 - 12\sqrt{2},$ then

$$\sum_{k=0}^{n} x^k H_{2k} = \frac{(2x - 34 + n(x-34))x^n H_{2n} + (n+1)x^n H_{2n-2} - 34}{2x - 34}.$$

(c) $(m = 2, j = 1)$

If $x^2 - 34x + 1 \neq 0,$ i.e., $x \neq 17 + 12\sqrt{2}, x \neq 17 - 12\sqrt{2},$ then

$$\sum_{k=0}^{n} x^k H_{2k+1} = \frac{(x-34)x^{n+1}H_{2n+1} + x^{n+1}H_{2n-1} - 6x + 6}{x^2 - 34x + 1},$$

and

if $x^2 - 34x + 1 = 0,$ i.e., $x = 17 + 12\sqrt{2}$ or $x = 17 - 12\sqrt{2},$ then

$$\sum_{k=0}^{n} x^k H_{2k+1} = \frac{((2+n)x - 34(n+1))x^n H_{2n+1} + (n+1)x^n H_{2n-1} - 6}{2x - 34}.$$

(d) $(m = -1, j = 0)$

If $x^2 - 6x + 1 \neq 0,$ i.e., $x \neq 3 + 2\sqrt{2}, x \neq 3 - 2\sqrt{2},$ then

$$\sum_{k=0}^{n} x^k H_{-k} = \frac{x^{n+1}H_{-n+1} + (x-6)x^{n+1}H_{-n} - 6x + 2}{x^2 - 6x + 1},$$

and

if $x^2 - 6x + 1 = 0,$ i.e., $x = 3 + 2\sqrt{2}$ or $x = 3 - 2\sqrt{2},$ then

$$\sum_{k=0}^{n} x^k H_{-k} = \frac{(n+1)x^n H_{-n+1} + (2x - 6 + n(x-6))x^n H_{-n} - 6}{2x - 6}.$$

(e) $(m=-2, j=0)$

If $x^2-34x+1\neq 0$, *i.e.,* $x\neq 17+12\sqrt{2}$, $x\neq 17-12\sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k H_{-2k} = \frac{x^{n+1}H_{-2n+2}+(x-34)x^{n+1}H_{-2n}-34x+2}{x^2-34x+1},$$

and

if $x^2-34x+1=0$, *i.e.,* $x=17+12\sqrt{2}$ *or* $x=17-12\sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k H_{-2k} = \frac{(n+1)x^n H_{-2n+2}+(2x-34+n(x-34))x^n H_{-2n}-34}{2x-34}.$$

(f) $(m=-2, j=1)$

If $x^2-34x+1\neq 0$, *i.e.,* $x\neq 17+12\sqrt{2}$, $x\neq 17-12\sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k H_{-2k+1} = \frac{x^{n+1}H_{-2n+3}+(x-34)x^{n+1}H_{-2n+1}-198x+6}{x^2-34x+1},$$

and

if $x^2-34x+1=0$, *i.e.,* $x=17+12\sqrt{2}$ *or* $x=17-12\sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k H_{-2k+1} = \frac{(n+1)x^n H_{-2n+3}+(2x-34+n(x-34))x^n H_{-2n+1}-198}{2x-34}.$$

From the above proposition, we have the following corollary which gives sum formulas of Lucas-balancing numbers (take $W_n = C_n$ with $C_0 = 1, C_1 = 3$).

Corollary 6.5. *For* $n\geq 0$, *Lucas-balancing numbers have the following properties:*

(a) $(m=1, j=0)$

If $x^2-6x+1\neq 0$, *i.e.,* $x\neq 3+2\sqrt{2}$, $x\neq 3-2\sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k C_k = \frac{(x-6)x^{n+1}C_n+x^{n+1}C_{n-1}-3x+1}{x^2-6x+1},$$

and

if $x^2-6x+1=0$, *i.e.,* $x=3+2\sqrt{2}$ *or* $x=3-2\sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k C_k = \frac{(2x-6+n(x-6))x^n C_n+(n+1)x^n C_{n-1}-3}{2x-6}.$$

(b) $(m=2, j=0)$

If $x^2-34x+1\neq 0$, *i.e.,* $x\neq 17+12\sqrt{2}$, $x\neq 17-12\sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k C_{2k} = \frac{(x-34)x^{n+1}C_{2n}+x^{n+1}C_{2n-2}-17x+1}{x^2-34x+1},$$

and

if $x^2-34x+1=0$, *i.e.,* $x=17+12\sqrt{2}$ *or* $x=17-12\sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k C_{2k} = \frac{(2x-34+n(x-34))x^n C_{2n}+(n+1)x^n C_{2n-2}-17}{2x-34}.$$

(c) $(m = 2, j = 1)$

If $x^2 - 34x + 1 \neq 0$, *i.e.,* $x \neq 17 + 12\sqrt{2}$, $x \neq 17 - 12\sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k C_{2k+1} = \frac{(x-34)x^{n+1}C_{2n+1} + x^{n+1}C_{2n-1} - 3x + 3}{x^2 - 34x + 1},$$

and

if $x^2 - 34x + 1 = 0$, *i.e.,* $x = 17 + 12\sqrt{2}$ *or* $x = 17 - 12\sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k C_{2k+1} = \frac{((2+n)x - 34(n+1))x^n C_{2n+1} + (n+1)x^n C_{2n-1} - 3}{2x - 34}.$$

(d) $(m = -1, j = 0)$

If $x^2 - 6x + 1 \neq 0$, *i.e.,* $x \neq 3 + 2\sqrt{2}$, $x \neq 3 - 2\sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k C_{-k} = \frac{x^{n+1}C_{-n+1} + (x-6)x^{n+1}C_{-n} - 3x + 1}{x^2 - 6x + 1},$$

and

if $x^2 - 6x + 1 = 0$, *i.e.,* $x = 3 + 2\sqrt{2}$ *or* $x = 3 - 2\sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k C_{-k} = \frac{(n+1)x^n C_{-n+1} + (2x - 6 + n(x-6))x^n C_{-n} - 3}{2x - 6}.$$

(e) $(m = -2, j = 0)$

If $x^2 - 34x + 1 \neq 0$, *i.e.,* $x \neq 17 + 12\sqrt{2}$, $x \neq 17 - 12\sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k C_{-2k} = \frac{x^{n+1}C_{-2n+2} + (x-34)x^{n+1}C_{-2n} - 17x + 1}{x^2 - 34x + 1},$$

and

if $x^2 - 34x + 1 = 0$, *i.e.,* $x = 17 + 12\sqrt{2}$ *or* $x = 17 - 12\sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k C_{-2k} = \frac{(n+1)x^n C_{-2n+2} + (2x - 34 + n(x-34))x^n C_{-2n} - 17}{2x - 34}.$$

(f) $(m = -2, j = 1)$

If $x^2 - 34x + 1 \neq 0$, *i.e.,* $x \neq 17 + 12\sqrt{2}$, $x \neq 17 - 12\sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k C_{-2k+1} = \frac{x^{n+1}C_{-2n+3} + (x-34)x^{n+1}C_{-2n+1} - 99x + 3}{x^2 - 34x + 1},$$

and

if $x^2 - 34x + 1 = 0$, *i.e.,* $x = 17 + 12\sqrt{2}$ *or* $x = 17 - 12\sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k C_{-2k+1} = \frac{(n+1)x^n C_{-2n+3} + (2x - 34 + n(x-34))x^n C_{-2n+1} - 99}{2x - 34}.$$

Taking $x = 1$ in the last three corollaries we get the following corollary.

Corollary 6.6. *For* $n \geq 0$, *balancing numbers, modified Lucas-balancing numbers and Lucas-balancing numbers have the following properties:*

1.

(a) $\sum_{k=0}^{n} B_k = \frac{1}{4}(5B_n - B_{n-1} - 1)$.
(b) $\sum_{k=0}^{n} B_{2k} = \frac{1}{32}(33B_{2n} - B_{2n-2} - 6)$.
(c) $\sum_{k=0}^{n} B_{2k+1} = \frac{1}{32}(33B_{2n+1} - B_{2n-1} - 2)$.
(d) $\sum_{k=0}^{n} B_{-k} = \frac{1}{4}(-B_{-n+1} + 5B_{-n} + 1)$.
(e) $\sum_{k=0}^{n} B_{-2k} = \frac{1}{32}(-B_{-2n+2} + 33B_{-2n} + 6)$.
(f) $\sum_{k=0}^{n} B_{-2k+1} = \frac{1}{32}(-B_{-2n+3} + 33B_{-2n+1} + 34)$.

2.

(a) $\sum_{k=0}^{n} H_k = \frac{1}{4}(5H_n - H_{n-1} + 4)$.
(b) $\sum_{k=0}^{n} H_{2k} = \frac{1}{32}(33H_{2n} - H_{2n-2} + 32)$.
(c) $\sum_{k=0}^{n} H_{2k+1} = \frac{1}{32}(33H_{2n+1} - H_{2n-1})$.
(d) $\sum_{k=0}^{n} H_{-k} = \frac{1}{4}(-H_{-n+1} + 5H_{-n} + 4)$.
(e) $\sum_{k=0}^{n} H_{-2k} = \frac{1}{32}(-H_{-2n+2} + 33H_{-2n} + 32)$.
(f) $\sum_{k=0}^{n} H_{-2k+1} = \frac{1}{32}(-H_{-2n+3} + 33H_{-2n+1} + 192)$.

3.

(a) $\sum_{k=0}^{n} C_k = \frac{1}{4}(5C_n - C_{n-1} + 2)$.
(b) $\sum_{k=0}^{n} C_{2k} = \frac{1}{32}(33C_{2n} - C_{2n-2} + 16)$.
(c) $\sum_{k=0}^{n} C_{2k+1} = \frac{1}{32}(33C_{2n+1} - C_{2n-1})$.
(d) $\sum_{k=0}^{n} C_{-k} = \frac{1}{4}(-C_{-n+1} + 5C_{-n} + 2)$.
(e) $\sum_{k=0}^{n} C_{-2k} = \frac{1}{32}(-C_{-2n+2} + 33C_{-2n} + 16)$.
(f) $\sum_{k=0}^{n} C_{-2k+1} = \frac{1}{32}(-C_{-2n+3} + 33C_{-2n+1} + 96)$.

7 MATRICES RELATED WITH GENERALIZED BALANCING NUMBERS

We define the square matrix A of order 2 as:

$$A = \begin{pmatrix} 6 & -1 \\ 1 & 0 \end{pmatrix}$$

such that $\det A = 1$. Then, we have

$$\begin{pmatrix} W_{n+1} \\ W_n \end{pmatrix} = \begin{pmatrix} 6 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} W_n \\ W_{n-1} \end{pmatrix} \tag{7.1}$$

and

$$\begin{pmatrix} W_{n+1} \\ W_n \end{pmatrix} = \begin{pmatrix} 6 & -1 \\ 1 & 0 \end{pmatrix}^n \begin{pmatrix} W_1 \\ W_0 \end{pmatrix}.$$

If we take $W_n = B_n$ in (7.1) we have

$$\begin{pmatrix} B_{n+1} \\ B_n \end{pmatrix} = \begin{pmatrix} 6 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} B_n \\ B_{n-1} \end{pmatrix}. \tag{7.2}$$

We also define

$$B_n = \begin{pmatrix} B_{n+1} & -B_n \\ B_n & -B_{n-1} \end{pmatrix}$$

and

$$C_n = \begin{pmatrix} W_{n+1} & -W_n \\ W_n & -W_{n-1} \end{pmatrix}.$$

Theorem 7.1. *For all integers $m, n,$ we have*

(a) $B_n = A^n$

(b) $C_1 A^n = A^n C_1$

(c) $C_{n+m} = C_n B_m = B_m C_n.$

Proof. Take $r = 6, s = -1$ in Soykan [28, Theorem 5.1.]. □

Corollary 7.2. *For all integers $n,$ we have the following formulas for the balancing, modified Lucas-balancing and Lucas-balancing numbers.*

(a) *balancing Numbers.*
$$A^n = \begin{pmatrix} 6 & -1 \\ 1 & 0 \end{pmatrix}^n = \begin{pmatrix} B_{n+1} & -B_n \\ B_n & -B_{n-1} \end{pmatrix}.$$

(b) *Modified Lucas-balancing Numbers.*
$$A^n = \begin{pmatrix} 6 & -1 \\ 1 & 0 \end{pmatrix}^n = \frac{1}{16} \begin{pmatrix} 3H_{n+1} - H_n & -(H_{n+1} - 3H_n) \\ H_{n+1} - 3H_n & -(H_n - 3H_{n-1}) \end{pmatrix}.$$

(c) *Lucas-balancing Numbers.*
$$A^n = \begin{pmatrix} 6 & -1 \\ 1 & 0 \end{pmatrix}^n = \frac{1}{8} \begin{pmatrix} 3C_{n+1} - C_n & -(C_{n+1} - 3C_n) \\ C_{n+1} - 3C_n & -(C_n - 3C_{n-1}) \end{pmatrix}.$$

Proof.

(a) It is given in Theorem 7.1 (a).

(b) Note that, from Lemma 4.4, we have

$$32B_n = 2H_{n+1} - 6H_n.$$

Using the last equation and (a), we get required result.

(c) Note that, from Lemma 4.5, we have

$$8B_n = C_{n+1} - 3C_n.$$

Using the last equation and (a), we get required result. □

Theorem 7.3. *For all integers $m, n,$ we have*

$$W_{n+m} = W_n B_{m+1} - W_{n-1} B_m \tag{7.3}$$

Proof. Take $r = 6, s = -1$ in Soykan [28, Theorem 5.2.]. □

By Lemma 4.1, we know that

$$(W_0^2 + W_1^2 - 6W_0W_1)B_n = -W_0W_{n+1} + W_1W_n,$$

so (7.3) can be written in the following form

$$(W_0^2 + W_1^2 - 6W_0W_1)W_{n+m} = W_n(-W_0W_{m+2} + W_1W_{m+1}) - W_{n-1}(-W_0W_{m+1} + W_1W_m).$$

Corollary 7.4. *For all integers $m, n,$ we have*

$$\begin{aligned} B_{n+m} &= B_n B_{m+1} - B_{n-1} B_m, \\ H_{n+m} &= H_n B_{m+1} - H_{n-1} B_m, \\ C_{n+m} &= C_n B_{m+1} - C_{n-1} B_m, \end{aligned}$$

and

$$8C_{n+m} = C_n(C_{m+2} - 3C_{m+1}) - C_{n-1}(C_{m+1} - 3C_m).$$

8 CONCLUSION

In the literature, there have been so many studies of the sequences of numbers and the sequences of numbers were widely used in many research areas, such as physics, engineering, architecture, nature and art. In this paper, we obtain some fundamental properties of generalized balancing numbers. We can summarize the sections as follows:

- In section 1, we give some background about generalized Fibonacci numbers and present a short history of balancing numbers.
- In section 2, we define generalized balancing sequence and then the generating functions and the Binet formulas have been given.
- In section 3, Simson formula of generalized balancing numbers are presented.
- In section 4, we obtain some identities of generalized balancing, balancing, modified Lucas-balancing and Lucas-balancing numbers.
- In section 5, we consider generalized balancing sequence at negative indices and construct the relationship between the sequence and itself at positive indices. This illustrates the recurrence property of the sequence at the negative index. Meanwhile, this connection holds for all integers.
- In section 6, we have written sum identities in terms of the generalized balancing sequence, and then we have presented the formulas as special cases the corresponding identity for the balancing, modified Lucas-balancing and Lucas-balancing sequences. All the listed identities in the proposition and corollaries may be proved by induction, but that method of proof gives no clue about their discovery. We give the proofs to indicate how these identities, in general, were discovered.
- In section 7, we give matrices related with these sequences (generalized balancing, balancing, modified Lucas-balancing and Lucas-balancing sequences).

COMPETING INTERESTS

Author has declared that no competing interests exist.

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