On $G^2$-Property of $G$-Metric Spaces

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ABSTRACT

The purpose of this paper is to introduce and investigate weak form of G-open sets in G-metric spaces, namely $G^2$-open sets. The relationships among this form with the other known sets are introduced. We give the notions of the interior operator, the closure operator and frontier operator via $G^2$-open sets.

Keywords: Open set; Metric spaces.

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1 INTRODUCTION

The concept of a metric space was introduced by Frechet in 1906, [1]. It has a very important basic role in mathematics and its application. Many mathematical concepts that can be discussed in this space. The first attempt to generalize the ordinary distance function to a distance of three points was introduced by Gahler, [2, 3], in 1993.

K. S. Ha, et al; [4], showed that a 2-metric is not a generalization of the usual notion of a metric. It was mentioned by Gähler, [2], that the notion of a 2-metric is an extension of an idea of ordinary metric and geometrically $(x, y, z)$ represents the area of a triangle formed by the points $x,y$ and $z$ in X as its vertices. But this is not always true.A.Sharma, [5], showed that $(x, y, z) = 0$ for any three distinct points $x, y, z \in \mathbb{R}^2$. B. C. Dhage

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in 1963 introduced a new class of generalized metrics called D-metrics, [3]. However, several errors for fundamental topological properties in a D-metric space were found by Z. Mustafa and B. Sims, [6]. Due to these considerations, Z. Mustafa and B. Sims [7] proposed a more appropriate notion of a generalized metric space, called G-metric space.

This paper is organized as follows. Section 2 is devoted to some preliminaries. Section 3 introduces the concept of Gβ-open sets by utilizing the G-open balls. Furthermore, the relationship with the other known sets will be studied. In Section 4 we introduce the concepts of the interior operator, the closure operator and frontier operator via Gβ-open sets.

2 PRELIMINARIES

Definition 2.1. [1] Let $X$ be any nonempty set. A function $d : X \times X \to [0, \infty)$ is called a metric function on $X$ if it satisfies the following three conditions for all $x, y, z \in X$:
1. (positive property) $d(x, y) \geq 0$ with equality if and only if $x = y$;
2. (symmetric property) $d(x, y) = d(y, x)$;
3. (triangle inequality) $d(x, z) \leq d(x, y) + d(y, z)$.

A pair $(X, d)$, where $d$ is a metric on $X$ is called a metric space.

Definition 2.2. [6] Let $X$ be a nonempty set and $\mathbb{R}$ be the set of real numbers. A function $G : X \times X \times X \to \mathbb{R}$ is called a G-metric function on $X$ if it satisfies the following:
1. $G(x, x, y) > 0$ for all $x \neq y \in X$;
2. $G(x, y, z) = 0$ if and only if $x = y = z$;
3. $G(x, y, z) \leq G(x, y, y)$ for every $x, y, z \in X$ with $y \neq z$;
4. $G(x, y, z) = G(p(x, y, z))$ for every $x, y, z \in X$ and for any permutation $p$ of $x, y, z$;
5. $G(x, y, z) \leq G(x, u, u) + G(u, y, z)$ for every $x, y, z, u \in X$.

If $G$ is a G-metric function on $X$, then the pair $(X, G)$ is called a G-metric space.

Example 2.3. [7] Let $(\mathbb{R}, d)$ be the usual metric space. Define $G$, by $G(x, y, z) = d(x, y) + d(y, z) + d(x, z)$ for all $x, y, z \in \mathbb{R}$. Then it is clear that $(\mathbb{R}, G)$ is a G-metric space.

Example 2.4. [7] Let $X = \{a, b\}$. Define $G$ on $X \times X \times X$ by $G(a, a, a) = G(b, b, b) = 0$, $G(a, a, b) = 1$, $G(a, b, b) = 2$.

Example 2.5. [7] Let $(\mathbb{R}, G)$ be a G-metric space defined by $G(x, y, z) = \max\{|x - y|, |y - z|, |z - x|\}$.

Definition 2.6. [8] Let $(X, G)$ be a G-metric space, $x \in X$ and $A \subseteq X$. The open ball with center $x$ and radius $\epsilon$ in metric space $(X, G)$ is denoted by $B_G(x, \epsilon)$ and defined by
$$B_G(x, \epsilon) = \{y \in X | d(x, y) < \epsilon\}.$$ 

The closed ball with center $x$ and radius $\epsilon$ in G-metric space $(X, G)$ is denoted by $C_G(x, \epsilon)$ and defined by
$$C_G(x, \epsilon) = \{y \in X | d(x, y) \leq \epsilon\}.$$ 

The set $A$ is called an open set in G-metric space $(X, G)$ if for every $x \in A$, there is $\epsilon > 0$ such that $B_G(x, \epsilon) \subseteq A$. The set $A$ is called closed set in metric space $(X, G)$ if $X - A$ is an open set in G-metric space $(X, G)$.

Theorem 2.7. [8] Every G-open ball $B_G(x, \epsilon), x \in X, \epsilon > 0$ is an open set in $X$. 

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Theorem 2.8. [7] Let \((X, G)\) be a G-metric space, then for any \(x \in X\) and \(\epsilon > 0\), we have.
(1) If \(G(y, x, x) < \epsilon\) then \(x, y \in B_G(x, \epsilon)\);
(2) If \(y \in B_G(x, \epsilon)\) then there exists a \(\delta > 0\) such that \(B_G(y, \delta) \subseteq B_G(x, \epsilon)\).

Definition 2.9. \([8]\) \(Cl_G(A)\) is called the \(G\)-closure of \(A\) if it is the intersection of all \(G\)-closed sets containing \(A\).

Definition 2.10. \([8]\) A set \(U\) in a G-metric space \(X\), is said to be closed if its complement \(X - U\) is \(G\)-open.

3 \(G^3\)-OPEN SETS

Definition 3.1. Let \((X, G)\) be a G-metric space and \(A \subseteq X\). A point \(x \in X\) is called a \(G\)-point of \(A\) in G-metric space \((X, G)\) if there is \(\delta > 0\) such that for every \(y \in B_G(x, \delta)\),

\[B_G(y, \epsilon) \cap G \neq \emptyset \quad \forall \epsilon > 0.\]

\(G^3(A)\) denotes the set of all \(G^3\)-points of \(A\) in G-metric space \((X, G)\)

Example 3.2. Let \((\mathbb{R}, G)\) be G-metric space defined by \(G(x, y, z) = \max\{|x - y|, |y - z|, |z - x|\}\). Let \(A = (0, 2)\) and \(B = Q\) be that set of rational numbers. Note that \(G^3(A) = (0, 2)\) and \(G^3(B) = \mathbb{R}\).

Theorem 3.3. Let \((X, G)\) be any G-metric space and \(A, B \subseteq X\). Then
1. \(G^3(\phi) = \phi\) and \(G^3(X) = X\);
2. if \(A \subseteq B\) Then \(G^3(A) \subseteq G^3(B)\);
3. \(G^3(A \cap B) \subseteq G^3(A) \cap G^3(B)\);
4. \(G^3(A) \cup G^3(B) \subseteq G^3(A \cup B)\).

Proof. 1. It is clear from the definition, we get that \(G^3(\phi) = \phi\) and \(G^3(X) = X\).
2. Let \(A \subseteq B\) and \(x \in G^3(A)\). Then is \(\delta > 0\) such that for every \(y \in B_G(y, \epsilon) \cap A \neq \emptyset\), for all Since \(A \subseteq B\). Then \(B_G(y, \epsilon) \cap B \neq \emptyset\), for all \(\epsilon > 0\). That is, \(x \in G^3(B)\). Then \(G^3(A) \subseteq G^3(B)\).
3. Since \(A \cap B \subseteq A\). Then by part (2) \(G^3(A \cap B) \subseteq G^3(A)\). Similar \(G^3(A \cap B) \subseteq G^3(B)\) Then \(G^3(A \cap B) \subseteq G^3(A) \cap G^3(B)\).
4. Since \(A \subseteq (A \cup B)\). Then by part (2) \(G^3(A) \subseteq G^3(A \cup B)\). Similar \(G^3(A) \subseteq G^3(A \cup B)\) Then \(G^3(A) \cup G^3(B) \subseteq G^3(A \cup B)\).

Definition 3.4. Let \((X, G)\) be a G-metric space. A subset \(A \subseteq X\) is called a \(G^3\)-open set in G-metric space \((X, G)\) if for every \(x \in A\),

\[B_G(x, \epsilon) \cap G^3(A) \neq \emptyset \quad \forall \epsilon > 0.\]

A subset \(A \subseteq X\) is called a \(G^3\)-closed set in G-metric space \((X, G)\) if \(X - A\) is a \(G^3\)-open set in G-metric space \((X, G)\).

Example 3.5. In Example(3.2), the sets \(A\) and \(B\) are \(G^3\)-open sets. Note that any finite sub sets of \(\mathbb{R}\) are not \(G^3\)-open set.

Theorem 3.6. Every \(G\)-open set is a \(G^3\)-open set.
Let the union of any family of $G$-open sets be $G$-open set. That is, $B_C(y, \varepsilon) \cap G \neq \emptyset$ for every $\varepsilon > 0$. Hence $A$ is $G^3$-open set.

The converse of above theorem need not be true.

**Example 3.7.** In Example (3.2), note that for the closed interval $A = [a, b]$, $G^3(A) = (a, b)$. Then it is clear to check that $A$ is a $G^3$-open set. Take $x = a$ or $x = b$. Note that $x \in A$ but there is no $G$-open ball with center $x$ contained in $A$. That is, $A$ is not $G$-open set in $(\mathbb{R}, G)$.

The intersection of two $G^3$-open sets need not to be $G^3$-open set. In Example (3.2), set of rational numbers $Q$ is a $G^3$-open set but not $G$-open set in $(\mathbb{R}, G)$ and the set $IR \cup \{q\}$ is a $G^3$-open set in $(\mathbb{R}, G)$, where $IR$ is the set of irrational numbers and $q$ is any rational number, but $Q \cap (IR \cup \{q\}) = \{q\}$ is not $G^3$-open set. That is, the collection of all $G^3$-open sets in G-metric space $(X, G)$ does not form topology on a set $X$.

The following theorem shows that the intersection of a $G$-open set and a $G^3$-open set is a $G^3$-open set.

**Theorem 3.8.** The intersection of a $G$-open set and a $G^3$-open set is a $G^3$-open set.

**Proof.** Let $A$ be $G$-open set and $B$ be $G^3$-open set in G-metric space in $(X, G)$. Let $x \in A \cap B$ be arbitrary point. Then there is $\delta_1 > 0$ and $\delta_2 > 0$ such that $B_C(x, \delta_1) \subseteq A$ and for every $y \in B_C(x, \delta_2)$, $B_C(y, \varepsilon) \cap B \neq \emptyset$ for every $\varepsilon > 0$. Take $\delta = \min\{\delta_1, \delta_2\} > 0$. Then $B_C(x, \delta) \subseteq A$ and for every $y \in B_C(x, \delta)$, $B_C(y, \varepsilon) \cap B \neq \emptyset$ for every $\varepsilon > 0$. Now for every $y \in B_C(x, \delta)$ and since $A$ is $G$-open set, then there is $\varepsilon_y > 0$ such that $B_C(y, \varepsilon_y) \subseteq A$ and $B_C(y, \min\{\delta_1, \delta_2\}) \cap B \neq \emptyset$. Since $B_C(y, \min\{\delta_1, \delta_2\}) \cap B \subseteq B_C(y, \varepsilon) \cap A \cap B$, then $B_C(y, \varepsilon) \cap (A \cap B) \neq \emptyset$ for every $\varepsilon > 0$. That is $A \cap B$ is $G^3$-open set. Hence $x \in G^3(A \cap B)$. Then $B_C(y, \varepsilon) \cap G^3(A \cap B) \neq \emptyset$ for all $\varepsilon > 0$. There for $A \cap B$ is $G^3$-open set.

**Theorem 3.9.** The union of any family of $G^3$-open sets is $G^3$-open set.

**Proof.** Let $H_\lambda$ be a $G^3$-open in G-metric space $(X, G)$ for all $\lambda \in \Delta$. Let $x \in \cup_{\lambda \in \Delta} H_\lambda$ be an arbitrary point. Then there is at least $\lambda_0 \in \Delta$ such that $x \in H_{\lambda_0}$. Since $H_{\lambda_0}$ is a $G^3$-open set then $B_C(x, \varepsilon) \cap G^3(H_{\lambda_0}) \neq \emptyset$ for all $\varepsilon > 0$. Hence by Theorem (3.3), $G^3(H_{\lambda_0}) \subseteq G^3(\cup_{\lambda \in \Delta} H_\lambda)$. Hence $B_C(x, \varepsilon) \cap G^3(\cup_{\lambda \in \Delta} H_\lambda) \neq \emptyset$ for all $\varepsilon > 0$. That is $\cup_{\lambda \in \Delta} H_\lambda$ is $G^3$-open set.

## 4 $G^3$-OPEN OPERATORS

In this section, we define the interior operator, the closure operator and frontier operator via $G^3$-open sets.

**Definition 4.1.** Let $(X, G)$ be a G-metric space and $A \subseteq X$. The $G$-closure operator of $A$ is denoted by $Cl^G_3(A)$ and defined by

$$Cl^G_3(A) = \cap \{H \subseteq X : A \subseteq H \text{ and } H \text{ is } G^3\text{-closed set}\}.$$ 

The $G$-interior functor of $A$ is denoted by $Int^G_3(A)$ and defined by

$$Int^G_3(A) = \cup \{H \subseteq X : H \subseteq A \text{ and } H \text{ is } G^3\text{-open set}\}.$$
Remark 4.2.
1. By Theorem(3.9), $Cl^G(A)$ is a $G^3$-closed set and $Int^G(A)$ is $G^3$-open set in G-metric space $(X, G)$.
2. For a G-metric space $(X, G)$ and $A \subseteq X$, it is clear from the definition of $Cl^G(A)$ and $Int^G(A)$ that $A \subseteq Cl^G(A)$ and $Int^G(A) \subseteq A$.

Theorem 4.3. For a G-metric space $(X, G)$ and $A \subseteq X$, $Cl^G(A) = A$ if and only if $A$ is a $G^3$-closed set.

Proof. Let $Cl^G(A) = A$. Then from definition of $Cl^G(A)$ and Theorem(3.9), $Cl^G(A)$ is a $G^3$-closed set and $A$ is a $G^3$-closed set. Conversely, we have $A \subseteq Cl^G(A)$ by Remark(4.2). Since $A$ is a $G^3$-closed set, then it is clear from the definition of $Cl^G(A)$, $Cl^G(A) \subseteq A$. Hence $A = Cl^G(A)$. \hfill $\square$

Theorem 4.4. For a G-metric space $(X, G)$ and $A \subseteq X$, and $Int^G(A) = A$ if and only if $A$ is a $G^3$-open set.

Proof. Let $A$ be $G^3$-open set. Then for all $x \in A$, we have $x \in A \subseteq A$. That is, $A \subseteq Int^G(A)$. Then $A = Int^G(A)$ from Remark(4.2). The converse is trivial. \hfill $\square$

Theorem 4.5. For a G-metric space $(X, G)$ and $A \subseteq X$, $x \in Cl^G(A)$ if and only if for all $G^3$-open set $B$ containing $x$, $B \cap A \neq \emptyset$.

Proof. Let $x \in Cl^G(A)$ and $B$ be any $G^3$-open set containing $x$. If $B \cap A = \emptyset$ then $A \subseteq X - B$. Since $X - B$ is a $G^3$-closed set containing $A$, then $Cl^G(A) \subseteq X - B$ and so $x \in Cl^G(A) \subseteq X - B$. Hence this is contradiction, because $x \in B$. Therefore $B \cap A \neq \emptyset$.

Conversely, Let $x \notin Cl^G(A)$. Then $X - Cl^G(A)$ is a $G$-open set containing $x$. Hence by hypothesis, $[X - Cl^G(A)] \cap A \neq \emptyset$. But this is contradiction, because $X - Cl^G(A) \subseteq X - A$. \hfill $\square$

Theorem 4.6. For a G-metric space $(X, G)$ and $A \subseteq X$, $x \in Int^G(A)$ if and only if there is a $G^3$-open set $B$ such that $x \in B \subseteq A$.

Proof. Let $x \in Int^G(A)$ and take $B = Int^G(A)$. Then by Theorem(4.5) and definition of $Int^G(A)$ we get that $B$ is a $G^3$-open set and by Remark(4.2), $x \in B \subseteq A$. Conversely, let there is a $G^3$-open set $B$ such that $x \in B \subseteq A$ Then by definition of $Int^G(A)$, $x \in B \subseteq Int^G(A)$. \hfill $\square$

Theorem 4.7. For a G-metric space $(X, G)$ and $A, B \subseteq X$, the following hold:

1. If $A \subseteq B$ then $Cl^G(A) \subseteq Cl^G(B)$;
2. $Cl^G(A) \cup Cl^G(B) \subseteq Cl^G(A \cup B)$;
3. $Cl^G(A \cap B) \subseteq Cl^G(A) \cap Cl^G(B)$;
4. $Cl^G(A) \subseteq Cl_G(A)$.

Proof. 1. Let $x \in Cl^G(A)$. Then by Theorem(4.5), for all $G^3$-open set $C$ containing $x$, $C \cap A \neq \emptyset$. Since $A \subseteq B$ then $C \cap B \neq \emptyset$. Hence $x \in Cl^G(B)$. That is, $Cl^G(A) \subseteq Cl^G(B)$.

2. Since $A \subseteq A \cup B$ and $B \subseteq A \cup B$, then by part(1), $Cl^G(A) \subseteq Cl^G(A \cup B)$ and $Cl^G(B) \subseteq Cl^G(A \cup B)$. Hence $Cl^G(A \cup B) \subseteq Cl^G(A) \cup Cl^G(B)$.\hfill $\square$

3. Since $A \cap B \subseteq A$ and $B \cap B \subseteq B$, then by part(1), $Cl^G(A \cap B) \subseteq Cl^G(A)$ and $Cl^G(A \cap B) \subseteq Cl^G(B)$. Hence $Cl^G(A \cap B) \subseteq Cl^G(A) \cap Cl^G(B)$.\hfill $\square$

4. It is clear from Theorem(4.5) and from every G-open set is $G^3$-open set. \hfill $\square$

In the above theorem $Cl^G(A \cup B) \neq Cl^G(A) \cup Cl^G(B)$ as it is shown in the following example.
Example 4.8. Let \((\mathbb{R}, G)\) be \(G\)-metric space, where
\[ G(x, y, z) = \max\{ |x - y|, |y - z|, |z - x| \} \]
and \((\mathbb{R}, d)\) is usual metric space. Let \(A = IR\) and \(B = Q - \{2\}\), where \(Q\) is the set of rational numbers, \(IR\) is the set of irrational numbers and 2 is any rational number. Since \(A\) and \(B\) are \(G\alpha\)-closed sets in \(\mathbb{R}\). Then \(Cl^\alpha_G(A) \cup Cl^\alpha_G(B) = A \cup B = \mathbb{R} - \{2\}\). If \(\mathbb{R} - \{2\}\) is \(G\beta\)-closed set in \(\mathbb{R}\) then \(\{2\}\) is \(G\beta\)-open set but \(\{2\}\) is not \(G\beta\)-open set and this contradiction. Hence \(\mathbb{R} - \{2\}\) is not \(G\alpha\)-closed set in \(\mathbb{R}\). Since \(\mathbb{R} - \{2\}\subseteq Cl^\alpha_G(\mathbb{R} - \{2\})\) then
\[ Cl^\alpha_G(A \cup B) = Cl^\alpha_G(\mathbb{R} - \{2\}) = R. \]

Theorem 4.9. For a \(G\)-metric space \((X, G)\) and \(A, B \subseteq X\), the following hold:
1. If \(A \subseteq B\) then \(Int^\alpha_G(A) \subseteq Int^\alpha_G(B)\);
2. \(Int^\alpha_G(A) \cup Int^\alpha_G(B) \subseteq Int^\alpha_G(A \cup B)\);
3. \(Int^\alpha_G(A \cap B) \subseteq Int^\alpha_G(B) \cap Int^\alpha_G(B)\);
4. \(Int^\alpha_G(A) \subseteq Int^\alpha_G(A)\).

Proof. 1. Let \(x \in Int^\alpha_G(A)\). Then by Theorem(4.6), there is a \(G\beta\)-open set \(C\) such that \(x \in C \subseteq A\). Since \(A \subseteq B\) then \(x \in C \subseteq B\). Hence \(x \in Int^\alpha_G(B)\). That is, \(Int^\alpha_G(A) \subseteq Int^\alpha_G(B)\).
2. Since \(A \subseteq A \cup B\) and \(B \subseteq A \cup B\), then by part(1), \(Int^\alpha_G(A) \subseteq Int^\alpha_G(A \cup B)\) and \(Int^\alpha_G(B) \subseteq Int^\alpha_G(A \cup B)\). Hence \(Int^\alpha_G(A) \cup Int^\alpha_G(B) \subseteq Int^\alpha_G(A \cup B)\).
3. Since \(A \cap B \subseteq A\) and \(A \cap B \subseteq B\), then by part(1), \(Int^\alpha_G(A \cap B) \subseteq Int^\alpha_G(B)\) and \(Int^\alpha_G(A \cap B) \subseteq Int^\alpha_G(A)\). Hence \(Int^\alpha_G(A \cap B) \subseteq Int^\alpha_G(A) \cap Int^\alpha_G(B)\).
4. It is clear from Theorem(4.5) and from every \(G\)-open set is \(G\alpha\)-open set.

In the last theorem \(Int^\alpha_G(A \cap B) \neq Int^\alpha_G(A) \cap Int^\alpha_G(B)\) as it is shown in the following example.

Example 4.10. In Example(4.8), take \(A = Q \cup \{\sqrt{2}\}\) and \(B = IR\), where \(Q\) is the set of rational numbers, \(IR\) is the set of irrational numbers and \(\sqrt{2}\) is any irrational number. Since \(A\) and \(B\) are \(G\alpha\)-open sets in \(\mathbb{R}\). Then \(Int^\alpha_G(A) \cap Int^\alpha_G(B) = A \cap B = (Q \cup \{\sqrt{2}\}) \cap IR = \{\sqrt{2}\}\). Since \(\{\sqrt{2}\}\) is not \(G\beta\)-open set and \(Int^\alpha_G(\{\sqrt{2}\}) \subseteq \{\sqrt{2}\}\) then \(Int^\alpha_G(A \cap B) = Int^\alpha_G(\{\sqrt{2}\}) = \emptyset\).

Theorem 4.11. For a \(G\)-metric space \((X, G)\) and \(G \subseteq X\), the following hold:
1. \(Int^\alpha_G(X - A) = X - Cl^\alpha_G(A)\);
2. \(Cl^\alpha_G(X - A) = X - Int^\alpha_G(A)\).

Proof. 1. Since \(A \subseteq Cl^\alpha_G(A)\), then \(X - Cl^\alpha_G(A) \subseteq X - A\). Since \(Cl^\alpha_G(A)\) is a \(G\beta\)-closed set then \(X - Cl^\alpha_G(A)\) is a \(G\beta\)-open set. Then
\[ X - Cl^\alpha_G(A) = Int^\alpha_G(X - Cl^\alpha_G(A)) \subseteq Int^\alpha_G(X - A). \]

For the other side, let \(x \in Int^\alpha_G(X - A)\). Then there is \(G\beta\)-open set \(C\) such that \(x \in C \subseteq X - A\). Then \(X - C\) is a \(G\beta\)-closed set containing \(A\) and \(x \notin X - C\). Hence \(x \notin Cl^\alpha_G(G)\), that is, \(x \notin X - Cl^\alpha_G(A)\).
2. Since \(Int^\alpha_G(A) \subseteq A\), then \(X - A \subseteq X - Int^\alpha_G(A)\). Since \(Int^\alpha_G(A)\) is a \(G\beta\)-open set then \(X - Int^\alpha_G(A)\) is a \(G\beta\)-closed set. Then
\[ Cl^\alpha_G(X - A) = Cl^\alpha_G(X - Int^\alpha_G(A)) = X - Int^\alpha_G(A). \]

For the other side, let \(x \notin Cl^\alpha_G(X - A)\). Then by Theorem(4.5), there is a \(G\beta\)-open set \(C\) containing \(x\) such that \(C \cap (X - A) = \emptyset\). Then \(x \in C \subseteq A\), that is, \(x \in Int^\alpha_G(A)\). Hence \(x \notin X - Int^\alpha_G(A)\). Therefore \(X - Int^\alpha_G(A) \subseteq Cl^\alpha_G(X - A)\).
Theorem 4.12. For a subset $A \subseteq X$ of G-metric space $(X, G)$ the following hold:

1. If $B$ is a G-open set in $X$ then $\text{Cl}_G^\beta (A) \cap B \subseteq \text{Cl}_G^\beta (A \cap B)$;
2. If $B$ is a G-closed set in $X$ then $\text{Int}_G^\beta (A \cup B) \subseteq \text{Int}_G^\beta (A) \cup B$.

Proof. 1. Let $x \in \text{Cl}_G^\beta (A) \cap B$. Then $x \in \text{Cl}_G^\beta (A)$ and $x \in B$. Let $D$ be any $G^\beta$-open set in $(X, G)$ containing $x$. By Theorem(3.8), $D \cap B$ is $G^\beta$-open set containing $x$. Since $x \in \text{Cl}_G^\beta (A)$ then by Theorem(4.5), $(D \cap B) \cap A \neq \emptyset$. This implies, $D \cap (B \cap A) \neq \emptyset$. Hence by Theorem(4.5), $x \in \text{Cl}_G^\beta (A \cap B)$. That is, $\text{Cl}_G^\beta (A) \cap B \subseteq \text{Cl}_G^\beta (A \cap B)$.

2. Since $B$ is a G-closed set $X$ then by the part(1) and Theorem(4.11),

$$X - [\text{Int}_G^\beta (A) \cup B] = [X - \text{Int}_G^\beta (A)] \cap [X - B] \subseteq \text{Cl}_G^\beta ([X - A] \cap (X - B)) = \text{Cl}_G^\beta (X - (A \cup B)) = X - (\text{Int}_G^\beta (A \cup B)).$$

Hence $\text{Int}_G^\beta (A \cup B) \subseteq \text{Int}_G^\beta (A) \cup B$. □

Theorem 4.13. For a G-metric space $(X, G)$ and $A \subseteq X$, $x \in \text{Cl}_G(A)$ if and only if for all $\varepsilon > 0$, $B_G(x, \varepsilon) \cap A \neq \emptyset$.

Proof. Let $x \in \text{Cl}_G(A)$ and $\varepsilon > 0$. If $B_G(x, \varepsilon) \cap A = \emptyset$ then $A \subseteq X - B_G(x, \varepsilon)$. Since $X - B_G(x, \varepsilon)$ is a G-closed set containing $A$, then $\text{Cl}_G(A) \subseteq X - B_G(x, \varepsilon)$ and $x \in \text{Cl}_G(A) \subseteq X - B_G(x, \varepsilon)$. Hence this is contradiction, because $x \in B_G(x, \varepsilon)$. Therefore $B_G(x, \varepsilon) \cap A \neq \emptyset$.

Conversely, Let $x \notin \text{Cl}_G(A)$. Then $X - \text{Cl}_G(A)$ is a G-open set containing $x$. Then there is $\varepsilon > 0$ such that $B_G(x, \varepsilon) \subseteq X - \text{Cl}_G(A)$ Hence by hypothesis, $B_G(x, \varepsilon) \cap A \neq \emptyset$. But this is contradiction, because $B_G(x, \varepsilon) \subseteq X - \text{Cl}_G(A) \subseteq X - A$. □

For a subset $A$ of G-metric space $(X, G)$ the G-frontier operator of $A$ is defined by

$$\Gamma_G^\beta (A) = \text{Cl}_G^\beta (A) - \text{Int}_G^\beta (A).$$

Theorem 4.14. For a subset $A \subseteq X$ of G-metric space $(X, G)$, the following hold:

1. $\text{Cl}_G^\beta (A) = \Gamma_G^\beta (A) \cup \text{Int}_G^\beta (A)$;
2. $\Gamma_G^\beta (A) \cap \text{Int}_G^\beta (A) = \emptyset$;
3. $\Gamma_G^\beta (A) = \text{Cl}_G^\beta (A) \cap \text{Cl}_G^\beta (X - A)$.

Proof. 1. Note that

$$\Gamma_G^\beta (A) \cup \text{Int}_G^\beta (A) = (\text{Cl}_G^\beta (A) - \text{Int}_G^\beta (A)) \cup \text{Int}_G^\beta (A) = [\text{Cl}_G^\beta (A) \cap (X - \text{Int}_G^\beta (A))] \cup \text{Int}_G^\beta (A) = [\text{Cl}_G^\beta (A) \cup \text{Int}_G^\beta (A)] \cap [(X - \text{Int}_G^\beta (A)) \cup \text{Int}_G^\beta (A)] = \text{Cl}_G^\beta (A) \cap X = \text{Cl}_G^\beta (A).$$

2. It is clear from the definition of $\Gamma_G^\beta (A)$.
3. By Theorem(4.11),

$$\Gamma_G^\beta (A) = \text{Cl}_G^\beta (A) - \text{Int}_G^\beta (A) = \text{Cl}_G^\beta (A) \cap (X - \text{Int}_G^\beta (A)) = \text{Cl}_G^\beta (A) \cap \text{Cl}_G^\beta (X - A).$$
Corollary 4.15. For a subset \( A \subseteq X \) of G-metric space \((X, G)\), \( \Gamma^G_G(A) \) is \(G^G\)-closed set in \((X, G)\).

Proof. By Theorem(4.9) and the part(3) of the last theorem. 

Theorem 4.16. For a subset \( A \subseteq X \) of G-metric space \((X, G)\), the following hold:

1. \( A \) is a \( G^G \)-open set if and only if \( \Gamma^G_G(A) \cap A = \emptyset \);
2. \( A \) is a \( G^G \)-closed set if and only if \( \Gamma^G_G(A) \subseteq A \);
3. \( A \) is both \( G^G \)-open set and \( G^G \)-closed set if and only if \( \Gamma^G_G(A) = \emptyset \).

Proof. 1. Let \( A \) be a \( G^G \)-open set. Then \( \text{Int}^G_G(A) = A \). Then by Theorem(4.14),

\[
\Gamma^G_G(A) \cap A = \Gamma^G_G(A) \cap \text{Int}^G_G(A) = \emptyset.
\]

Conversely, suppose that \( \Gamma^G_G(A) \cap A = \emptyset \). Then

\[
A - \text{Int}^G_G(A) = [A \cap \text{Cl}^G_G(A)] - [A \cap \text{Int}^G_G(A)] = A \cap (\text{Cl}^G_G(A) - \text{Int}^G_G(A)) = A \cap \Gamma^G_G(A) = \emptyset.
\]

That is, \( \text{Int}^G_G(A) = A \). Hence \( A \) is a \( G^G \)-open set.

2. Let \( A \) be a \( G^G \)-closed set. Then \( \text{Cl}^G_G(A) = A \). Then

\[
\Gamma^G_G(A) = \text{Cl}^G_G(A) - \text{Int}^G_G(A) = A - \text{Int}^G_G(A) \subseteq A.
\]

Conversely, suppose that \( \Gamma^G_G(A) \subseteq A \). Then by Theorem(4.14),

\[
\text{Cl}^G_G(A) = \text{Int}^G_G(A) \cup \Gamma^G_G(A) \subseteq \text{Int}^G_G(A) \cup A \subseteq A.
\]

That is, \( \text{Cl}^G_G(A) = A \). Hence \( A \) is \( G^G \)-closed set.

3. Let \( A \) be both \( G^G \)-closed set and \( G^G \)-open set. Then \( \text{Cl}^G_G(A) = A = \text{Int}^G_G(A) \). Then

\[
\Gamma^G_G(A) = \text{Cl}^G_G(A) - \text{Int}^G_G(A) = A - A = \emptyset.
\]

Conversely, suppose that \( \Gamma^G_G(A) = \emptyset \). Then \( \text{Cl}^G_G(A) - \text{Int}^G_G(A) = \emptyset \). Since \( \text{Int}^G_G(A) \subseteq A \subseteq \text{Cl}^G_G(A) \) then \( \text{Cl}^G_G(A) = \text{Int}^G_G(A) \). Since \( \text{Int}^G_G(A) \subseteq A \subseteq \text{Cl}^G_G(A) \) then

\[
\text{Cl}^G_G(A) = A = \text{Int}^G_G(A).
\]

That is, \( \text{Cl}^G_G(A) = A \). Hence \( A \) is both \( G^G \)-closed set and \( G^G \)-open set.

\[\square\]

5 CONCLUSION

As we noted that the \( G^G \)-open set is a weak form of open set in G-metric space, also the reader can give the notion of the continty property via \( G^G \)-open sets in G-metric spaces. The reader also can introduce seperation axioms connectedness and compactness properties by using \( G^G \)-open sets in G-metric spaces.

COMPETING INTERESTS

Authors have declared that no competing interests exist.

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